LTI REPRESENTATIONS OF ADAPTIVE SYSTEMS WITH TAP DELAY-LINE REGRESSORS UNDER SINUSOIDAL EXCITATION

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Abstract

It is shown that an adaptive system whose regressor is formed by tap delay-line (TDL) filtering of a multitone sinusoidal signal is representable as a parallel connection of an linear time-invariant (LTI) block and a linear time-varying (LTV) block. Furthermore, a norm-bound (induced 2-norm) is computed explicitly on the LTV block and is shown to decrease as N^{-1} where N is the number of taps. Hence it follows that the adaptive system becomes LTI in the limit as the number of taps goes to infinity. In the more realistic case of finite N, the model can be systematically analyzed and designed using modern robust control methods applicable to LTI systems with norm-bounded perturbations. 'I'his analysis extends Glover's 1977 results by putting them into a modern robust control perspective, and allowing precise statements to be made about the stability of the system in the presence of the LTV subsystem,

1 INTRODUCTION

In 1977, Glover [3] [10] established an important result that an adaptive feedforward controller whose regressor is formed by tap delay-line (TDL) filtering of a multitone sinusoidal signal can be written as the parallel connection of a linear time-invariant (LTI) and linear time-varying (LTV) subsystem. Glover then makes a heuristic argument that for a large number of taps N the LTI subsystem will dominate the LTV subsystem. This provides a useful approximation because the LTI subsystem has the transfer function of a resonance filter which forms a frequency notch when applied in closed-loop. This provides an important alternative interpretation of the adaptive notching effect associated with the adaptive controller, For example, the LTI interpretation provides estimates of the transients, the depth of the frequency notches, the closed-loop pole locations, etc. all of which are not available from using standard Lyapunov and Hyperstability methods alone [6].

Rigorously speaking, Glover's LTI analysis of TDL regressors is only applicable in the limiting case of an infinite number of taps $N = \infty$. It ignores the contribution of the LTV subsystem for a finite number of taps, and does not provide any characterization of the convergence properties as the number of taps increases. Hence, for the realistic case of a finite number of taps, even the stability properties of Glover's analysis are in question. 'l'he present paper overcomes these limitations by putting this problem into a modern robust control setting, Specifically, a norm-bound (the induced 2-norm) is established on the LTV subsystem which is shown to converge to zero at a particular rate as a function of the number of taps, the adaptation gain, the number of tones, and the tone spacing. With this representation, for any finite N, the adaptive system can be rigorously analyzed using robust control methods applicable to LTI systems with norm-bounded perturbations, The choice of N can be determined from precise H_∞ conditions rather than the heuristic "rules of thumb" given in Glover's original paper,

2 BACKGROUND

2.1 Adaptive Systems with Harmonic Regressors

The configuration to be studied is shown in Figure 2.1. An estimate $\hat{y}$ of some signal y is to be constructed as a linear combination of the elements of a regressor vector $x(t) \in R^N$, i.e.,

Estimated Signal

$$\hat{y} = w(t)^T x(t) \tag{2.1}$$

where $w(t) \in RN$ is a parameter vector which is tuned in real-time using the adaptation algorithm, *Adaptation Algorithm*

$$w = \mu\Gamma(p)[\tilde{x}(t)e(t)] \tag{2.2}$$

Here, the notation $\Gamma(p)[\cdot]$ is used to denote the multivariable LT1 transfer function $\Gamma(s)$,1 where $\Gamma(s)$ is any LTI transfer function in the Laplace s operator (the differential operator *p* will replace the Laplace operator s in all time-domain filtering expressions); the term e(t) $\in R^1$ is an error signal; $\mu > 0$ is an adaptation gain; and the signal $\tilde{x}$ is obtained by filtering the regressor x through any stable filter *F'(p),* i.e.,

Regressor *Filtering*

$$\tilde{x} = F(p)[x] \tag{2.3}$$

The notation $F(p)[\cdot]$ denotes the multivariable LTI transfer function $F(s)\cdot I$ with S1S0 filter F'(s), acting on the indicated vector time domain signal.

For the purposes of this paper, it will be assumed that the regressor x can be written as a linear combination of m distinct sinusoidal components $\{\omega_i\}_{i=1}^{m}$, O $< \omega_1 < \omega_2 < \ldots < \omega_m$. Equivalently, it is assumed that there exists a matrix $\mathcal{X} \in R^{Nx2m}$ such that,

Harmonic Regressor

$$x = \mathcal{X}c(t) \tag{2.4}$$

$$c(t) = [\sin(\omega_1 t), \cos(\omega_1 t), \ldots \sin(\omega_m t), \cos(\omega_m t)]^T \in R?^{m} \tag{2.5}$$

Equations (2.1)-(2,5) taken together will be referred to as a *harmonic adaptive system.* Collectively, these equations define an important open-loop mapping from the error signal c to the estimated output $\hat{y}$. Because of its importance, this mapping will be denoted by the special character 7-t, i.e.,

$$\hat{y} = \mathcal{H}[e] \tag{2.6}$$

The special structure of 7-t is depicted in Figure 2,1,

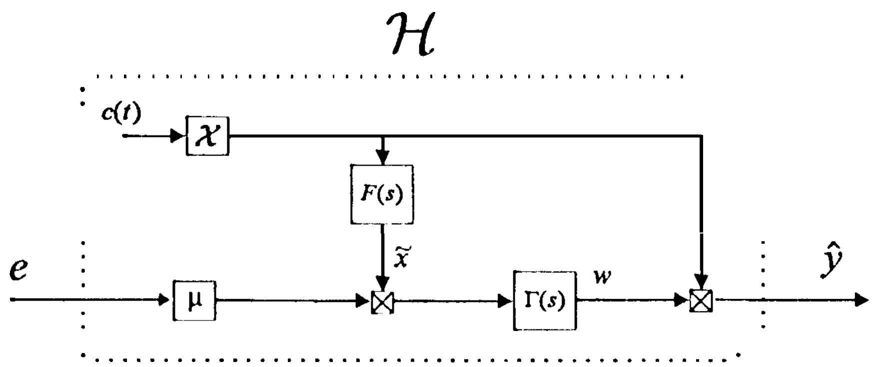

Figure 2.1: LTV operator $\hat{y} = \mathcal{H}[e]$ for adaptive system with harmonic regressor x, adaptation law $\Gamma(s)$, and regressor filter $F(s)$

REMARK 2,1 The definition of I'(s) is left intentionally general to include analysis of the gradient algorithm (i.e., with the choice $\Gamma(s) = 1/s$), the gradient algorithm with leakage (i.e., I'(s) = $1/(s+\sigma); \sigma \geq$ O), proportional-plus-integral adaptation (i.e., $\Gamma(s) = k_p + k_i/s$), or arbitrary linear adaptation algorithms of the designer's choosing. " Adaptation laws which are nonlinear or normalized (e.g., divided by the norm of the regressor), are not considered here since they do not have an equivalent LTI representation $\Gamma(p)$. ∎

REMARK 2.2 The use of the regressor filter $F(s)$ is (2,3) allows the unified treatment of many important adaptation algorithms including the well-known Filtered-X algorithm from the signal processing literature [8] [5] [2] [9], and the Augmented Error algorithm of Monopoli [4], Since x is comprised purely of sinusoidal components and F in **(2.3) is** stable, all subsequent analysis will assume that the filter output $\tilde{x}$ has reached a steady-state condition. ∎

The following result taken from [1] will bc needed which gives necessary and sufficient conditions for the operator 7-t to be LTI.

THEOREM 2.1 (LTI **Representation Theorem)** *Let the regressor $x(t)$ in the adaptive system (2,1) -(2. S) be given by the general multitone harmonic expression* ***(2.4)(2.5)*** *where the frequencies $\{\omega_i\}_{i=1}^m$ are distinct, nonzero, and $|F(j\omega_i)| > 0$ for all i.*

Then,

(i) *The mapping $\mathcal{H}$ from e to $\hat{y}$ is exactly representable as the linear time-invariant operator,*

$$\mathcal{H}: \quad \hat{y} = \bar{H}(p)c \tag{2.7}$$

if and only if the matrix $\mathcal{X}$ in **(2.4)** *satisfies the following X-Orthogonality (XO) condition,*

$\mathcal{X}$-Orthogonality (XO) Condition:

$$\mathcal{X}^T \mathcal{X} = D^2 \tag{2.8}$$

$$D^2 \triangleq \begin{bmatrix} {d_1}^2 \cdot I_{2\times2} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_m^2 \cdot I_{2\times2} \end{bmatrix} \in R^{2m\times2m} \tag{2.9}$$

where, $d_i^2 \geq 0, i$ = 1 , ..., m are scalars and $I_{2\times2} \in R^{2\times2}$ is the matrix identity.

(ii) $\overline{H}(s)$ *in (2.7) is given in closed- form-, as,*

$$\overline{H}(s) = \mu \sum_{i=1}^{m} d_i{}^2 \cdot H_i(s) \tag{2.10}$$

$$H_i(s) = \frac{F_R(i)}{2}\Big(\Gamma(s - j\omega_i) + \Gamma(s + j\omega_i)\Big) + \frac{F_I(i)}{2j}\Big(\Gamma(s - j\omega_i) - \Gamma(s + j\omega_i)\Big) \tag{2.11}$$

$$F_R(i) \triangleq \mathbf{Re}(F(j\omega_i)); \quad F_I(i) \triangleq \mathbf{Im}(F(j\omega_i)) \tag{2.12}$$

■

DEFINITION 2.1 *The matrix* $\mathcal{X}^T\mathcal{X} = D^2$ *having the special pairwise diagonal structure (2.9) in Theorem 2.1* is *defined as the* **confluence matrix** associated *with a particular harmonic adaptive system (2.1)-(2.5).* ■

The following result taken from [I.] shows that in the general case where the XO condition is not satisfied, the mapping $\mathcal{H}$ can always be decomposed into a *parallel connection* of an LTI subsystem and an LTV perturbation.

'THEOREM 2.2 **(LTI/LTV Decomposition)** Consider *the adaptive system ('2.1)-(2.9) with harmonic regresso r (2.4)(2.5). Then,*

(i) *In general the mapping* $\mathcal{H}$ *from e to* $\hat{y}$ *can be expressed as the parallel connection of an LTI block* $\overline{H}(s)$*, and an LTV perturbation block* $\tilde{\Delta}$,

$$\mathcal{H}: \quad \hat{y} = \overline{H}(p)e + \tilde{\Delta}[e] \tag{2.13}$$

where,

$$\overline{H}(s) \triangleq \mu \sum_{i=1}^{m} d_i{}^2 \cdot H_i(s) \tag{2.14}$$

$$\tilde{\Delta}[e] \triangleq \mu c(t)^T \Delta\Gamma(p)\left[\mathcal{F}c(t)e\right] \tag{2.15}$$

$$\Delta \triangleq \mathcal{X}^T\mathcal{X} - D^2 \tag{2.16}$$

$$\mathcal{F} \triangleq blockdiag\{\mathcal{F}_i\} \in R^{2m\times 2m} \tag{2.17}$$

$$\mathcal{F}_i \triangleq \begin{bmatrix} F_R(i) & F_I(i) \\ -F_I(i) & F_R(i) \end{bmatrix} \in R^{2\times 2}; \qquad for\ i = 1, \ldots, m \tag{2.18}$$

$$F_R(i) \triangleq \mathbf{Re}(F(j\omega_i)); \; F_I(i) \triangleq \mathbf{Im}(F(j\omega_i)) \tag{2.19}$$

and where Hi(s) is as defined in (2,11) of Theorem 2.1, and D^2 *is* chosen *(non-uniquely) as any matrix of the* 2×2 *block-diagonal form (2.9).*

(ii) *If the adaptation law $\Gamma(s)$ is stable with infinity norm $||\Gamma(s)||_\infty$, then the gain of the LTV perturbation can be bounded from above as,*

$$||\tilde{\Delta}||_{2i} \leq \mu m \bar{\sigma}(\Delta)||\Gamma(s)||_\infty \max_i |F(\omega_i)| \tag{2.20}$$

where $||\,,||_{2i}$ denotes the induced $\mathcal{L}_2$-norm of the indicated operator.

REMARK 2.3 The LTI/LTV decomposition in Theorem 2,2 is important for adaptive systems which do not exactly satisfy the XO condition. In this case, the adaptive system can be analyzed using modern robust control methods (i.e,, small gain theorem) making use of the analytic expression (2.14) for-the LTI block $\bar{H}(s)$ and the norm bound (2.20) on the time-varying perturbation block Δ [7] [1 1]. The induced $\mathcal{L}_2$-norm has been bounded here since it is consistent with the usc of H_∞ theory for robustness analysis. ∎

3 TAP DELAY-LINE (TDL) BASIS

3.1 Single Tone Case

In 1977, Glover [3] made the interesting discovery that an LTI system arises if the regressor x of the adaptive system is constructed by filtering a sinusoid through a long tap delay line (TDL). Glover's result can be understood simply in terms of the XO condition of Theorem 2.1.

Let the regressor $x(t) = [x_1(t), \ldots, x_N(t)]^T \in R^N$ be defined by filtering a *single* frequency $\omega_1 > 0$,

$$\xi(t) = a_{11}\sin(\omega_1 t) + a_{12}\cos(\omega_1 t) \tag{3.1}$$

through a TDL with N taps and tap delay T, i.e.,

$$x_\ell(t) = e^{-(\ell-1)pT}\xi(t), \ell = 1, \ldots, N \tag{3.2}$$

where the term $e^{-(\ell-1)pT}$ in the differential operator p represents a delay of $(\ell - 1)T$ time units. If $x(t)$ *is* written in the form $x = \mathcal{X}c(t)$, then it can be shown (i.e., set i = 1 and $m = 1$ in Theorem A.1 of Appendix A), that $\mathcal{X}$ satisfies,

$$\mathcal{X}^T\mathcal{X} = \frac{N}{2}\alpha_1^2 \cdot I_{2\times 2} + \frac{1}{2}\mathcal{B}_N(\omega_1 T)A_1^T R_{11} A_1 \tag{3.3}$$

where,

$$\alpha_1^2 = a_{11}^2 + a_{12}^2 \tag{3.4}$$

$$A_1 = \begin{bmatrix} a_{12} & -a_{11} \\ a_{11} & a_{12} \end{bmatrix}; \quad R_{11} = \begin{bmatrix} -\cos(N-1)\omega_1 T & \sin(N-1)\omega_1 T \\ \sin(N-1)\omega_1 T & \cos(N-1)\omega_1 T \end{bmatrix} \tag{3.5}$$

$$\mathcal{B}_N(\nu) \triangleq \frac{\sin N\nu}{\sin \nu} \tag{3.6}$$

The first term of (3.3) (a pairwise diagonal matrix), increases as N, while the second term remains bounded. Normalizing the adaptation gain to $\mu = \bar{\mu}/N$ for some $\bar{\mu} > 0$ (to prevent unbounded feedforward gain), and taking the limit as $N \to \infty$ gives,

$$\lim_{N\to\infty} \frac{1}{N}\mathcal{X}^T\mathcal{X} = \frac{1}{2}\alpha_1^2 \cdot I_{2\times 2} \tag{3.7}$$

Since the XO condition Theorem 2.1 is satisfied asymptotically as N gets large, the system becomes asymptotically LTI and is given as,

$$\mathcal{H}: \quad \hat{y} = \frac{\bar{\mu}}{2}{\alpha_1}^2 \cdot H_1(p)e \tag{3.8}$$

where H_1 (s) is given by (2.11)(2.12), and where $\alpha_1^2 = a_{11}^2 + a_{12}^2$. Restricting the choice of filter to $F(s) = 1$ in (3.8), and the choice of adaptation algorithm to the gradient algorithm (i.e., $\Gamma(s) = 1/s$) , gives precisely Glover's result (proved originally in the discrete-time case [3]). This result is recovered here as a special case of Theorem 2.1.

3.2 Multi-Tone Case

Rigorously speaking, Glover's LTI analysis of TDL regressors is only applicable in the limiting case when $N = \infty$, i.e., it ignores the contribution of the LTV subsystem for finite N, and does not provide any characterization of the convergence properties as N increases. In contrast, a more complete solution can be found by putting Glover's results into a modern robust control setting. This will be done in the present section by applying the LTI/LTV decomposition of Theorem 2.2 to the TDL regressor case. First, a definition will be useful.

DEFINITION 3.1 *Given time delay T and spacing parameter $0 < \underline{\nu} < \pi/2$, a Bounded Tone Set $\Omega(m, T, \underline{\nu})$ is defined as any set of m frequencies $\{\omega_i\}_{i=1}^m$, such that,*

$$\Omega(m,T,\underline{\nu}) \triangleq \Big\{ \quad \{\omega_i\}_{i=1}^m : \quad 0 < \underline{\nu} < \pi/2;$$

$$0 < \underline{\nu} < \omega_i T \leq \pi - \underline{\nu} \;\text{ for all } i = 1, \ldots, m;$$

$$|\omega_i - \omega_j|T \geq 2\underline{\nu} \qquad \text{for all } i \neq j \Big\} \tag{3.9}$$

■

Simply stated, a Bounded Tone Set is a set of frequencies $\{\omega_i\}_{i=1}^m$ which are bounded away from 0, π/T and each other. The definition is not very restrictive since any signal comprised of a finite number of distinct sinusoids lies in a Bounded Tone Set when T is chosen sufficiently small (i.e., to ensure Nyquist sampling of its highest component). Definition 3.1 is conveniently used to define the minimum *spacing* parameter $\underline{\nu}$ which will play an essential role in many subsequent results.

The main result of the paper follows.

THEOREM 3.1 **(Tap Delay-Line Basis)** *Consider the adaptive system (2.1)-(2.3) with harmonic regressor (2.4)(2.5), and input/output mapping $\mathcal{H}$ in (2.6). Let the components of the regressor $x = [x_1, \ldots, x_N]^T \in R^N$ be defined by filtering a signal $\xi(t) \in R^1$ through a tap delay line with N taps and tap delay T, i.e.,*

$$x_\ell = e^{-(\ell-1)pT}\xi, \; \ell = 1, \ldots, N \tag{3.10}$$

where the measured signal ξ is given by the following sum of m sinusoids,

$$\xi(t) = \sum_{i=1}^{m} \alpha_i \sin(\omega_i t + \phi_i); \quad \alpha_i > 0 \tag{3.11}$$

and frequencies $\{\omega_i\}_{i=1}^m$ lie in a bounded tone set $\Omega(m, T, \underline{\nu})$.

Then,

(i) *The regressor $x(t)$ can be written in harmonic form (2.4)(2.5) where the matrix $\mathcal{X} \in R^{N \times 2m}$ satisfies,*

$$\mathcal{X}^T\mathcal{X} = D^2 + \Delta \tag{3.12}$$

$$D^2 = \frac{N}{2}\begin{bmatrix} \alpha_1^2 \cdot I_{2\times2} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_m^2 \cdot I_{2\times2} \end{bmatrix} \in R^{2m\times2m} \tag{3.13}$$

and the matrix perturbation $\Delta \triangleq \mathcal{X}^T\mathcal{X} - D^2$ is norm-bounded as,

$$\bar{\sigma}(\Delta) \leq \frac{m\pi\alpha_{max}^2}{2\underline{\nu}}; \quad \alpha_{max} \triangleq \max_i\{\alpha_i\} \tag{3.14}$$

(ii) (LTI/LTV Decomposition)

The mapping $\mathcal{H}$ from $\hat{y}$ to e can be uniquely decomposed into the parallel connection of an LTI block $\bar{H}(s)$, and an LTV perturbation block $\tilde{\Delta}$,

$$\mathcal{H}: \quad \hat{y} = \bar{H}(p)e + \tilde{\Delta}[e] \tag{3.15}$$

where,

$$\bar{H}(s) \triangleq \mu\frac{N}{2}\sum_{i=1}^{m}\alpha_i^2 \cdot H_i(s) \tag{3.16}$$

$$\tilde{\Delta}[e] \triangleq \mu c(t)^T \Delta\Gamma(p)\left[\mathcal{F}c(t)e\right] \tag{3.17}$$

where the perturbation matrix A is *defined in (3.12?) with norm bound (3.14), and* $H_i(s)$ *is given by (2.11) of Theorem 2.1.*

Furthermore, if the adaptation law $\Gamma(s)$ *is stable with infinity norm* $||\Gamma(s)||_\infty$*, then the* gain *of the LTV perturbation can be bounded as,*

$$||\tilde{\Delta}||_{2i} \leq \frac{\mu m^2 \pi}{2\nu} \cdot \left(\alpha_{max}^2 ||\Gamma(s)||_\infty \max_i |F(\omega_i)|\right) \tag{3.18}$$

where $||\cdot||_{2i}$ *indicates the induced* $\mathcal{L}_2$*-norm.*

(iii) (Normalized Adaptation Gain)

By normalizing the adaptive gain to $\mu = \bar{\mu}/N$*, the operators in (9.16)(9. 17) of (ii) become,*

$$\bar{H}(s) = \frac{\bar{\mu}}{2}\sum_{i=1}^{m}\alpha_i^2 \cdot H_i(s) \tag{3.19}$$

$$\tilde{\Delta}[e] \triangleq \frac{\bar{\mu}}{N} c(t)^T \Delta\Gamma(p)\left[\mathcal{F}c(t)e\right] \tag{3.20}$$

and *the upper bound on the gain of the LTV perturbation in (3.18) becomes,*

$$||\tilde{\Delta}||_{2i} \leq \frac{\bar{\mu} m^2 \pi}{2N\nu} \cdot \left(\alpha_{max}^2 ||\Gamma(s)||_\infty \max_i |F(\omega_i)|\right) \tag{3.21}$$

where $||\Gamma(s)||_\infty$ is *assumed to exist.*

(iv) (Asymptotic Propertied)

If the adaptation law $\Gamma(s)$ *is stable (with bounded infinity norm* $||\Gamma(s)||_\infty$*), and the adaptation gain is normalized as* $\mu = \bar{\mu}/N$ *for* $\bar{\mu} > 0$ *constant, then as* $N \to \infty$ *the mapping* $\mathcal{H}$ *becomes LTI with asymptotic transfer function,*

$$\bar{H}(s) = \frac{\bar{\mu}}{2}\sum_{i=1}^{m}\alpha_i^2 \cdot H_i(s) \tag{3.22}$$

PROOF:

Proof of **(i):** From Theorem A.1 we have $\mathcal{X}^T\mathcal{X} = M$ where M is given by (A.52)-(A.55). Hence, $\Delta = M - D^2 \in R^{2m\times 2m}$ has the symmetric block 2-by-2 structure (A.77) used in Lemma A.5. Applying the result (A.78) of Lemma A.5 gives,

$$\bar{\sigma}(\Delta) \leq m \cdot \max_{i,j} \bar{\sigma}(\Delta_{ij}) \leq m \cdot \max_{i,j} \frac{\pi \alpha_i \alpha_j}{2\underline{\nu}} \tag{3.23}$$

$$\leq \frac{m\pi}{2\underline{\nu}} \max_i \{\alpha_i^2\} \triangleq \frac{m\pi}{2\underline{\nu}} \alpha_{max}^2 \tag{3.24}$$

where use has been made of property P.5 of Lemma A.4 in equation (3.23).

Proof of (ii): Results follow by applying the LTI/LTV Decomposition of Theorem 2.2 noting from (3. 13) that in the present case $d_i^2 = \frac{N}{2}\alpha_i^2$.

Proof of **(iii):** Simply substitute $\mu = \bar{\mu}/N$ into the LT1 and LTV blocks, where $\bar{\mu} > 0$ is a constant.

Proof of **(iv):** It is seen that the normalized LTI transfer function $\bar{H}(s)$ in (3.19) remains unaffected as N increases while the normalized LTV perturbation in (3.20) goes to zero as N increases. Hence, as N -i ∞ the mapping $\mathcal{H}$ becomes LTI with asymptotic transfer function given in (3.22), as desired. ●

For convenience, the results of Theorem 3.1 are summarized in Figure 3.1. Specifically, Figure 3.1 Part a. shows the harmonic adaptive system with TDL basis and normalized adaptation gain $\mu = \bar{\mu}/N$; Part b. shows the equivalent decomposition into an LTI block and a norm bounded LTV perturbation block. Note that the time-varying perturbation block goes to zero asymptotically as N becomes large.

REMARK 3.1 The asymptotic result (iv) of Theorem 3.1 follows essentially from the special form of the matrix $\mathcal{X}^T\mathcal{X}$ in (3,12) which arises in the TDL case. Specifically, in relation (3.12), the matrix D^2 given by (3.13) (and hence the associated LTI block) grows linearly with the number of taps N, while the perturbation matrix A (and hence the associated LTV block) remains *bounded* as N increases.

Hence, when the LTI and LTV paths are normalized by $1/N$ through choice of adaptation gain $\mu = \bar{\mu}/N$ (as shown in Figure 3.1) and the limit is taken as N becomes large, the LTI part remains constant while the norm bound on the LTV part decreases as $1/N$. This indicates that the LTV part can be made arbitrarily small by choosing N sufficiently large, while the LTI part remains unaffected. ■

REMARK 3.2 Interestingly, the bound on the LTV Perturbation $\tilde{\Delta}$ in (321) depends on the boundedness of the tone-set through the minimum spacing parameter $\underline{\nu} > 0$ defined in Definition 3.1, Specifically, a smaller $\underline{\nu}$ requires a larger N to justify the asymptotic

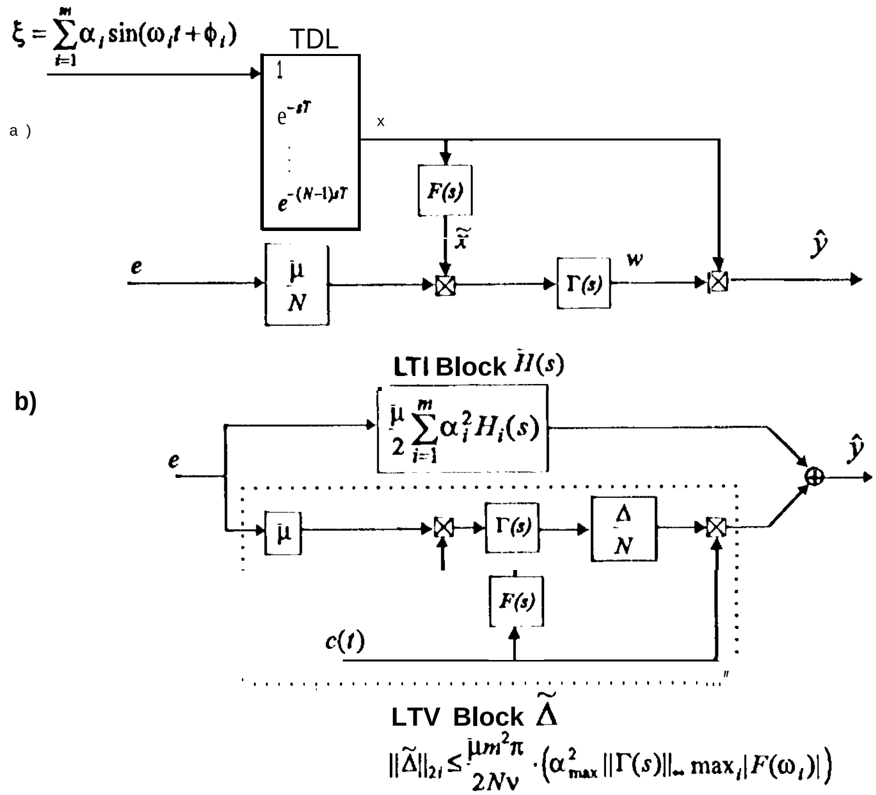

Figure 3.1: LTI/LTV decomposition of $\mathcal{H}$ for harmonic adaptive system with TDL basis

approximation to the same degree. This relationship precisely characterizes the effect of tone spacing on the convergence of the LTV subsystem, and settles a long standing question on the role of tone spacing in determining the asymptotic properties of adaptive systems with long tap delay lines. It is worth noting that these results are consistent with the heuristic discussions of tone spacing found Glover's original paper [3] (cf., Section IV, pp. 488).

Also interesting is the appearance of m^2 in the numerator of the norm bound (3.21). This indicates that if the number of tones m in the regressor is increased, one must increase N as the *square* of m to justify the asymptotic approximation to the same degree. This dependence on the tone count also appears to be new. ●

4 CONCLUSIONS

It has been shown that systems whose regressors are formed by filtering multitone sinusoidal signals through *tap delay-lines* satisfy the XO condition and hence have LTI representations in the limit as the number of taps becomes infinite, This result is of significant practical importance since tap delay lines, FIR filter representations, etc., are commonly used in many adaptive signal processing and communications applications,

For the more realistic case of a finite number of taps, Theorem 3.1 extends Glover's work by putting the problem into a modern robust control setting. Specifically, the unwanted time-varying terms (arising in Glover's expansion of the adaptive operator), are characterized precisely in terms of a norm bounded perturbation (3.21). This permits a rigorous treatment of the time-varying dynamics using modern robust control theory, and exposes the exact nature of the convergence to an LTI system as the number of taps is increased. The norm bound (3.21) is seen to be proportional to $m^2/(N\underline{\nu})$ which clearly indicates the size of the LTV perturbation as a function of the number of taps N, the minimum tone spacing parameter $\underline{\nu}$, and the number of tones m. To the author's knowledge, this is the first time these dependencies have been made explicit. Using this new model, the choice of N can be determined from precise H_∞ conditions rather than the heuristic "rules of thumb" given in Glover's original paper, and precise statements can be made about the stability of the system even in the presence of the LTV subsystem.

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A APPENDIX: Properties of TDL Regressors

The purpose of Appendix A is to provide the detailed structure of $\mathcal{X}^T\mathcal{X}$ for a tap delay-line basis (i.e., in Theorem A. I), and additional supporting results which are needed to prove Theorem 3.1.

The following definitions will be used throughout Appendix A:

$$C_i \triangleq \begin{bmatrix} 1 \\ \cos\omega_i T \\ \vdots \\ \cos(N-1)\omega_i T \end{bmatrix} \in R^N; \qquad S_i \triangleq \begin{bmatrix} 0 \\ \sin\omega_i T \\ \vdots \\ \sin(N-1)\omega_i T \end{bmatrix} \in R^N \tag{A1}$$

$$\underline{c}_{ij} \triangleq \cos((N-1)(\omega_i - \omega_j)T/2) \tag{A.2}$$

$$\underline{s}_{ij} \triangleq \sin((N-1)(\omega_i - \omega_j)T/2) \tag{A.3}$$

$$c_{ij} \triangleq \cos((N-1)(\omega_i + \omega_j)T/2) \tag{A.4}$$

$$s_{ij} \triangleq \sin((N-1)(\omega_i + \omega_j)T/2) \tag{A.5}$$

$$\underline{R}_{ij} \triangleq \begin{bmatrix} \underline{c}_{ij} & \underline{s}_{ij} \\ -\underline{s}_{ij} & \underline{c}_{ij} \end{bmatrix} \tag{A.6}$$

$$R_{ij} \triangleq \begin{bmatrix} -c_{ij} & s_{ij} \\ s_{ij} & c_{ij} \end{bmatrix} \tag{A.7}$$

$$\xi(t) \triangleq \sum_{i=1}^{m} \alpha_i \sin(\omega_i t + \phi_i) = \sum_{i=1}^{m} a_{i1}\sin(\omega_i t) + a_{i2}\cos(\omega_i t) \tag{A.8}$$

$$A_i \triangleq \begin{bmatrix} a_{i2} & -a_{i1} \\ a_{i1} & a_{i2} \end{bmatrix} \tag{A.9}$$

$$\mathcal{B}_N(\nu) \triangleq \frac{\sin N\nu}{\sin \nu} \tag{A.10}$$

LEMMA A1. *Let $\mathcal{B}_N(\nu)$ be defined by (A.10). Then, on the interval $0 \le \nu \le \pi$, the following inequality holds,*

$$|\mathcal{B}_N(\nu)| \le \frac{\pi}{2\tau(\nu)}; \quad \text{for } 0 \le \nu \le \pi \tag{A11}$$

where,

$$\tau(\nu) \triangleq \min(\nu, \pi - \nu) \tag{A.12}$$

PROOF: A sinusoid $\sin \nu$ can be bounded below on the interval $0 \le \nu \le \pi$ by piecewise linear segments as follows,

$$|\sin \nu| \ge \frac{1}{\pi} \min(2\nu, 2(\pi - \nu)); \quad \text{for } 0 \le \nu \le \pi \tag{A.13}$$

Hence,

$$\left|\frac{\sin N\nu}{\sin \nu}\right| \le \frac{\pi |\sin N\nu|}{\min(2\nu, 2(\pi - \nu))} \le \frac{\pi}{2 \cdot \tau(\nu)} \tag{A.14}$$

∎

LEMMA A.2 *Let $\mathcal{B}_N(\nu)$ be defined by (A,10). Then for frequencies $\{\omega_i\}_{i=1}^m$ in a bounded tone set $\Omega(m, T, \underline{\nu})$, the following inequalities hold,*

$$|\mathcal{B}_N(\omega_i T)| \le \frac{\pi}{2\underline{\nu}} \tag{A.15}$$

$$|\mathcal{B}_N((\omega_i - \omega_j)T/2)| \le \frac{\pi}{2\underline{\nu}} \tag{A.16}$$

$$|\mathcal{B}_N((\omega_i + \omega_j)T/2)| \le \frac{\pi}{2\nu} \tag{A.17}$$

PROOF: By properties of the Bounded Tone Set (3,9), the following inequalities can be shown to hold for any $\omega_i, \omega_j \in \Omega(m, T, \underline{\nu})$,

$$2\underline{\nu} \le |\omega_i - \omega_j| T \le \pi - 2\underline{\nu} \tag{A.18}$$

$$2\underline{\nu} \le |\omega_i + \omega_j| T \le 2\pi - 2\underline{\nu} \tag{A.19}$$

Proof of (A.15):

$$\mathcal{B}_N(\omega_i T) \le \frac{\pi}{2\tau(\omega_i T)} = \frac{\pi}{2\min(\omega_i T, \pi - \omega_i T)} \le \frac{\pi}{2\underline{\nu}} \tag{A.20}$$

Here, the first inequality in (A.20) follows by (A. 11) of Lemma A.1; the second equality follows by definition of τ in (A. 12); and the last inequality follows by properties of the bounded tone set $\Omega(m, T, \underline{\nu})$ in (3.9).

Proof of (A.16):

$$\mathcal{B}_N((\omega_i - \omega_j)T/2) \leq \frac{\pi}{2\tau((\omega_i - \omega_j)T/2)} \tag{A.21}$$

$$= \frac{\pi}{\min(|\omega_i - \omega_j|\,T, 2\pi - |\omega_i - \omega_j|\,T)} \tag{A.22}$$

$$\leq \frac{\pi}{\min(2\nu, 2\pi - (\pi - 2\nu))} = \frac{\pi}{2\nu} \tag{A.23}$$

Here, equation (A.21) follows by (A.11); (A.22) follows by the definition of $\tau(\cdot)$ in (A.12) and the fact that the function $\mathcal{B}_N(\cdot)$ *is* even; and (A.23) follows from (A.18),

Proof of (A.17):

$$\mathcal{B}_N((\omega_i + \omega_j)T/2) \leq \frac{\pi}{2\tau((\omega_i + \omega_j)T/2)} \tag{A.24}$$

$$= \frac{\pi}{\min(|\omega_i + \omega_j|T, 2\pi - |\omega_i + \omega_j|T)} \tag{A.25}$$

$$\leq \frac{\pi}{\min(2\nu, 2\pi - (2\pi - 2\nu))} = \frac{\pi}{2\nu} \tag{A.26}$$

Here, equation (A.24) follows by (A.11); (A.25) follows by the definition of $\tau(\cdot)$ in (A.12) and the fact that the function $\mathcal{B}_N(\cdot)$ is even; and (A.26) follows from (A.19), ■

LEMMA A.3 *Let C_i, S_i be as defined in (A.1). Then the following identities hold,*

$$S_i^T S_j = \frac{1}{2}\left(\underline{c}_{ij} \cdot \mathcal{B}_N\left((\omega_i - \omega_j)T/2\right) - c_{ij} \cdot \mathcal{B}_N\left((\omega_i + \omega_j)T/2\right)\right) \tag{A.27}$$

$$C_i^T C_j = \frac{1}{2}\left(\underline{c}_{ij} \cdot \mathcal{B}_N\left((\omega_i - \omega_j)T/2\right) + c_{ij} \cdot \mathcal{B}_N\left((\omega_i + \omega_j)T/2\right)\right) \tag{A.28}$$

$$C_j^T S_i = S_i^T C_j = \frac{1}{2}\left(s_{ij} \cdot \mathcal{B}_N\left((\omega_i + \omega_j)T/2\right) + \underline{s}_{ij} \cdot \mathcal{B}_N\left((\omega_i - \omega_j)T/2\right)\right) \tag{A.29}$$

$$C_i^T S_j = S_j^T C_i = \frac{1}{2}\left(s_{ij} \cdot \mathcal{B}_N\left((\omega_i + \omega_j)T/2\right) - \underline{s}_{ij} \cdot \mathcal{B}_N\left((\omega_i - \omega_j)T/2\right)\right) \tag{A.30}$$

where $\underline{c}_{ij}, \underline{s}_{ij}, s_{ij}, c_{ij}, \mathcal{B}_N$ *are as defined in (A.2)-(A.5), and (A.10),*

PROOF: Use will be made of the following identity,

$$\sum_{\ell=1}^{N} e^{j(\ell-1)\nu} = \frac{1 - e^{jN\nu}}{1 - e^{j\nu}} \tag{A.31}$$

$$= e^{j(N-1)\nu/2} \cdot \frac{\sin(N\nu/2)}{\sin(\nu/2)} \triangleq e^{j(N-1)\nu/2} \cdot \mathcal{B}_N(\nu/2) \tag{A.32}$$

Proof of (A.27):

$$S_i^T S_j = \sum_{\ell=1}^{N} \sin\Big(\omega_i(\ell-1)T\Big)\sin\Big(\omega_j(\ell-1)T\Big) \tag{A.33}$$

$$= \frac{1}{2}\sum_{\ell=1}^{N} \cos\Big((\omega_i-\omega_j)(\ell-1)T\Big) - \cos\Big((\omega_i+\omega_j)(\ell-1)T\Big) \tag{A.34}$$

$$= \frac{1}{2}\mathbf{Re}\left\{\sum_{\ell=1}^{N} e^{j(\omega_i-\omega_j)(\ell-1)T} - e^{j(\omega_i+\omega_j)(\ell-1)T}\right\} \tag{A.35}$$

$$= \frac{1}{2}\mathbf{Re}\Big\{e^{j(N-1)(\omega_i-\omega_j)T/2}\mathcal{B}_N((\omega_i-\omega_j)T/2)$$

$$- e^{j(N-1)(\omega_i+\omega_j)T}\mathcal{B}_N((\omega_i+\omega_j)T/2)\Big\} \tag{A.36}$$

$$= \frac{1}{2}\Big(\underline{c}_{ij}\mathcal{B}_N((\omega_i-\omega_j)T/2) - c_{ij}\mathcal{B}_N((\omega_i+\omega_j)T/2)\Big) \tag{A.37}$$

Here, $\mathbf{Re}(\cdot)$ denotes the real part of the indicated expression, Equations (A.33)-(A.35) follow by standard trig formulas; equation (A.36) follows by using identity (A.32); and (A.37) follows by the definition of s_{ij} and $\underline{s}_{ij}$ in (A.2)-(A.5).

Proof of (A.28):

$$C_i^T C_j = \sum_{\ell=1}^{N} \cos\Big(\omega_i(\ell-1)T\Big)\cos\Big(\omega_j(\ell-1)T\Big) \tag{A.38}$$

$$= \frac{1}{2}\sum_{\ell=1}^{N} \cos\Big((\omega_i-\omega_j)(\ell-1)T\Big) + \cos\Big((\omega_i+\omega_j)(\ell-1)T\Big) \tag{A.39}$$

$$= \frac{1}{2}\mathbf{Re}\left\{\sum_{\ell=1}^{N} e^{j(\omega_i-\omega_j)(\ell-1)T} + e^{j(\omega_i+\omega_j)(\ell-1)T}\right\} \tag{A.40}$$

$$= \frac{1}{2}\mathbf{Re}\Big\{e^{j(N-1)(\omega_i-\omega_j)T/2}\mathcal{B}_N((\omega_i-\omega_j)T/2) \tag{A.41}$$

$$+ e^{j(N-1)(\omega_i+\omega_j)T}\mathcal{B}_N((\omega_i+\omega_j)T/2)\Big\} \tag{A.42}$$

$$= \frac{1}{2}\Big(\underline{c}_{ij}\mathcal{B}_N((\omega_i-\omega_j)T/2) + c_{ij}\mathcal{B}_N((\omega_i+\omega_j)T/2)\Big) \tag{A.43}$$

Here, equations (A.38)-(A.40) follow by standard trig formulas; equation (A.42) follows by using identity (A.32); and (A.43) follows by the definition of c_{ij} and $\underline{c}_{ij}$ in (A.2)-(A.5).

Proof of (A.29):

$$C_j^T S_i - S_i^T C_j = \sum_{\ell=1}^{N} \sin\Big(\omega_i(\ell - 1)T\Big)\cos\Big(\omega_j(\ell - 1)T\Big) \tag{A.44}$$

$$= \frac{1}{2}\sum_{\ell=1}^{N} \sin\Big((\omega_i + \omega_j)(\ell - 1)T\Big) + \sin\Big((\omega_i - \omega_j)(\ell - 1)T\Big) \tag{A.45}$$

$$= \frac{1}{2}\mathrm{Im}\left\{\sum_{\ell=1}^{N} e^{j(\omega_i+\omega_j)(\ell-1)T} + e^{j(\omega_i-\omega_j)(\ell-1)T}\right) \tag{A.46}$$

$$= \frac{1}{2}\mathrm{Im}\Big\{ e^{j(N-1)(\omega_i+\omega_j)T/2}\mathcal{B}_N((\omega_i + \omega_j)T/2)$$

$$+ e^{j(N-1)(\omega_i-\omega_j)T}\mathcal{B}_N((\omega_i - \omega_j)T/2)\Big\} \tag{A.47}$$

$$= \frac{1}{2}\Big(s_{ij}\mathcal{B}_N((\omega_i + \omega_j)T/2) + \underline{s}_{ij}\mathcal{B}_N((\omega_i - \omega_j)T/2)\Big) \tag{A.48}$$

Here, **Ire(.)** denotes the imaginary part of the indicated expression. Equations (A ,44)-(A.46) follow by standard trig formulas; equation (A.47) follows by using identity (A.32); and (A.48) follows by the definition of s_{ij} and $\underline{s}_{ij}$ in (A.2)-(A.5).

Proof of (A.30): This relation follows by reversing the roles of i and j in the proof of (A.29), making use of the antisymmetric property of $\sin(-\theta) = -\sin(\theta)$ and symmetric property of $\mathcal{B}_N(-\nu) = \mathcal{B}_N(\nu)$. ∎

THEOREM A.1 **($\mathcal{X}^T\mathcal{X}$ for TDL Basis)** *Let the component of the regressor* $x = [x_1, ..., x_N]^T \in$ ***RN*** *be defined by filtering a signal* $\xi(t) \in R^1$ *through a tap delay line with N taps and tap delay T, i. e.,*

$$x_\ell = e^{-(\ell-1)sT}\xi, \ \ell = 1, ..., N \tag{A.49}$$

where the measured signal ξ is given by the following sum of m distinct sinusoids,

$$\xi(t) = \sum_{i=1}^{m} \alpha_i \sin(\omega_i t + \phi_i) = \sum_{i=1}^{m} a_{i1}\sin(\omega_i t) + a_{i2}\cos(\omega_i t) \tag{A.50}$$

Then, the regressor *$x(t)$ is of the harmonic form,*

$$x = xc(t) \tag{A.51}$$

where *c(t) is defined in [2,5) and the matrix $X \in R^{N\times 2m}$ satisfies,*

$$\mathcal{X}^T\mathcal{X} = \begin{bmatrix} \mathcal{M}_{11} & \dots & \mathcal{M}_{1m} \\ \vdots & \ddots & \vdots \\ \mathcal{M}_{m1} & \dots & \mathcal{M}_{mm} \end{bmatrix} \in R^{2m\times 2m} \tag{A.52}$$

$$\mathcal{M}_{ij} \in R^{2\times 2};\ i = 1, \dots, m;\ j = 1, \dots, m \tag{A.53}$$

$$\mathcal{M}_{ij} = \frac{1}{2}A_i^T\left(\mathcal{B}_N\left((\omega_i - \omega_j)T/2\right)R_{ij} + \mathcal{B}_N\left((\omega_i + \omega_j)T/2\right)R_{ij}\right)A_j \tag{A.54}$$

$$\mathcal{M}_{ii} = \frac{N}{2}\alpha_i^2 \cdot I_{2\times 2} + \frac{1}{2}\mathcal{B}_N(\omega_i T)A_i^T R_{ii} A_i \tag{A.55}$$

where definitions (A. 1)-(A. 10) have been used.

PROOF': Using standard trigonometric identities, the ℓth clement x_ℓ of the delayed regressor (A.49) can be expanded as,

$$x_\ell(t) = e^{-(\ell-1)sT}\xi = \sum_{i=1}^{m} a_{i1}\sin\Big(\omega_i(t-(\ell-1)\,T)\Big) + a_{i2}\cos\Big(\omega_i(t-(\ell-1)T)\Big) \tag{A.56}$$

$$\begin{aligned} &= \sum_{i=1}^{m} a_{i1}\Big(\cos(\omega_i(\ell-1)T)\sin\omega_i t - \sin(\omega_i(\ell-1)T)\cos\omega_i t\Big) \\ &+ a_{i2}\Big(\sin(\omega_i(\ell-1)T)\sin\omega_i t + \cos(\omega_i(\ell-1)T)\cos\omega_i t\Big) \end{aligned} \tag{A.57}$$

Using (A.57), the full regressor $x(t)$ earl be decomposed in terms of the vectors S_i and C_i in (A, 1) as follows,

$$x(t) = \left[a_{12}S_1 + a_{11}C_1, -a_{11}S_1 + a_{12}C_1, \dots, a_{m2}S_m + a_{m1}C_m, -a_{m1}S_m + a_{m2}C_m\right]c(t) \tag{A.58}$$

Equivalently, (in matrix notation),

$$x(t) = Q\mathcal{A}c(t) \tag{A.59}$$

where,

$$Q \triangleq [S_1, C_1, \dots, S_m, C_m] \in \boldsymbol{R}^{N\times 2m} \tag{A.60}$$

$$\mathcal{A} \triangleq \begin{bmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_m \end{bmatrix} \in R^{2m\times 2m} \tag{A.61}$$

and $A_i \in R^{2\times 2}$ *is* defined as in (A.9). Hence, by inspection of (A.51) and (A.59), the matrix X is given as,

$$\mathcal{X} = Q\mathcal{A} \tag{A.62}$$

Squaring up $\mathcal{X}$ and using (A.62) gives,

$$\mathcal{X}^T\mathcal{X} = \mathcal{A}^TQ^TQ\mathcal{A} \triangleq \mathcal{M} \tag{A.63}$$

From the structure of (A.60)-(A.62), the components blocks of M can be computed as,

$$\mathcal{M}_{ij} = A_i^T \begin{bmatrix} S_i^TS_j & S_i^TC_j \\ C_i^TS_j & C_i^TC_j \end{bmatrix} A_j \tag{A.64}$$

Result (A.54) follows by substituting the identities (A.27)-(A.30) of Lemma A.3 into (A.64) and simplifying using expressions (A. 1)-(A. 10). Result (A ,55) follows by setting $i=j$ in result (A.54) and simplifying by using the relation $\mathcal{B}_N(0) = N$. ■

LEMMA A.4 *Define,*

$$\Delta \triangleq \mathcal{M} - D^2 = \{\Delta_{ij}\} \in R^{2m\times 2m} \tag{A.65}$$

$$\Delta_{ii} \triangleq \mathcal{M}_{ii} - \frac{N}{2}\alpha_i^2 \cdot I_{2\times 2} \in R^{2\times 2} \tag{A.66}$$

$$\Delta_{ij} \triangleq \mathcal{M}_{ij} \in R^{2\times 2} \tag{A.67}$$

where D^2 is defined by in (3.13) of Theorem 3.1, and the matrix M and its submatrices $\mathcal{M}_{ij}$, $\mathcal{M}_{ii}$ are defined by (A.52)(A.55) Of Theorem A1.

Let the quantities $\alpha_i, A_i, \underline{R}_{ij}, R_{ij}$ be defined by (A. 6)-(A ,9), and assume that all frequencies $\{\omega_i\}_{i=1}^m$ are drawn from a bounded tone set i.e., $\Omega(m, T, \underline{\nu})$ defined in Definition 3.1. Then the following **properties** *hold,*

P1. $\bar{\sigma}(A_i) = \alpha_i$

P2. $\bar{\sigma}(\underline{R}_{ij}) = 1$

P3. $\bar{\sigma}(R_{ij}) = 1$

P4. $\bar{\sigma}(\Delta_{ii}) \leq \frac{\pi\alpha_i^2}{2\underline{\nu}}$

P5. $\bar{\sigma}(\Delta_{ij}) \leq \frac{\pi\alpha_i\alpha_j}{2\underline{\nu}}$

PROOF: It follows from the definition of ξ in (A.8), that the variables a_{i1}, a_{i2}, α_i are related as

$$\alpha_i^2 = a_{i1}^2 + a_{i2}^2 \tag{A.68}$$

For the proof, extensive use will be made of the definitions (A.2)-(A. 10). Continuing, Proof of P1:

$$\bar{\sigma}(A_i) = \lambda_{max}^{\frac{1}{2}}(A_i^T A_i) = \lambda_{max}^{\frac{1}{2}}\left[(a_{i1}^2 + a_{i2}^2)\cdot I\right] = \alpha_i \tag{A.69}$$

Proof of P2:

$$\bar{\sigma}(\underline{R}_{ij}) = \lambda_{max}^{\frac{1}{2}}(\underline{R}_{ij}^T \underline{R}_{ij}) = \lambda_{max}^{\frac{1}{2}}\left[(\underline{c}_{ij}^2 + \underline{s}_{ij}^2)\cdot I\right] = 1 \tag{A.70}$$

Proof of P3:

$$\bar{\sigma}(R_{ij}) = \lambda_{max}^{\frac{1}{2}}(R_{ij}^T R_{ij}) = \lambda_{max}^{\frac{1}{2}}\left[(c_{ij}^2 + s_{ij}^2)\cdot I\right] = 1 \tag{A.71}$$

Proof of P4:

$$\bar{\sigma}(\Delta_{ii}) = \bar{\sigma}(\mathcal{M}_{ii} - \frac{N}{2}\alpha_i \cdot I_{2\times 2}) = \bar{\sigma}\left(\frac{1}{2}\mathcal{B}_N(\omega_i T) A_i^T R_{ii} A_i\right) \tag{A.72}$$

$$\leq \frac{1}{2}|\mathcal{B}_N(\omega_i T)| \cdot \bar{\sigma}(A_i)^2 \leq \frac{\pi\alpha_i^2}{4\underline{\nu}} \tag{A.73}$$

where (A.72) follows by (A.55) of Theorem A. 1; equation (A.73) follows by P3 for $i = j$; and the last inequality follows by Lemma A.2 and property P1 proved above.

Proof of P5:

$$\begin{aligned} \bar{\sigma}(\Delta_{ij}) &= \bar{\sigma}\left(\frac{1}{2}\mathcal{B}_N\left((\omega_i - \omega_j)T/2\right) A_i^T \underline{R}_{ij} A_j + \mathcal{B}_N\left((\omega_i + \omega_j)T/2\right) A_i^T R_{ij} A_j\right) && \text{(A.74)} \\ &\leq \frac{1}{2}\bar{\sigma}(A_i)\cdot\bar{\sigma}(A_j)\Big(\left|\mathcal{B}_N\left((\omega_i - \omega_j)T/2\right)\right| \cdot \bar{\sigma}(\underline{R}_{ij}) \\ &+ \left|\mathcal{B}_N\left((\omega_i + \omega_j)T/2\right)\right| \cdot \bar{\sigma}(R_{ij})\Big) && \text{(A.75)} \\ &= \frac{\pi\alpha_i\alpha_j}{2\underline{\nu}} && \text{(A.76)} \end{aligned}$$

Here, (A.74) follows from (A.54) of Theorem A. 1 and (A.67); and (A.76) follows by Lemma A.2 and properties PI, P2, and P3 proved above,

∎

LEMMA A.5 *Let $X = XT \in R^{2m\times 2m}$ be a symmetric matrix partitioned into 2 x 2 blocks, i. e.,*

$$X = \begin{bmatrix} X_{11} & \dots & X_{1m} \\ \vdots & \ddots & \vdots \\ X_{m1} & \dots & X_{mm} \end{bmatrix} \in R^{2m\times 2m}; \quad X_{ij} \in R^{2\times 2} \tag{A.77}$$

Then,

$$\bar{\sigma}(X) \leq m \cdot \max_{ij} \bar{\sigma}(X_{ij}) \tag{A.78}$$

PROOF: Let $x \in R^{2m}$ be a vector partitioned compatibly with X, i.e., $x = [x_1^T, \ldots x_m^T]^T, x_i \in R^2$. It will be useful to define the matrix $\Gamma_k \in R^{2m \times 2m}$ which is the block diagonal matrix defined by,

$$\Gamma_k = diag\left[X_{k1}^T X_{k1}, \ldots . X_{km}^T X_{km}\right] \tag{A.79}$$

Consider the inequality,

$$(X_{ik}^T x_i - X_{kj} x_j)^T (X_{ik}^T x_i - X_{kj} x_j) \geq 0 \tag{A.80}$$

Expanding (A.80) and using symmetry $X_{ij}^T = X_{ji}$ gives upon rearranging,

$$x_i^T X_{ki}^T X_{ki} x_i + x_j^T X_{kj}^T X_{kj} x_j \geq 2 x_i^T X_{ik} X_{kj} x_j \tag{A.81}$$

Continuing,

$$\bar{\sigma}^2(X) \triangleq \max_{||x||=1} x^T X^T X x = \max_{||x||=1} \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} x_i^T X_{ik} X_{kj} x_j \tag{A.82}$$

$$\leq \frac{1}{2} \max_{||x||=1} \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} x_i^T X_{ki} X_{ki} x_i + x_j^T X_{kj} X_{kj} x_j \tag{A.83}$$

$$= m \max_{||x||=1} \sum_{j=1}^{m} \sum_{k=1}^{m} x_j^T X_{kj}^T X_{kj} x_j = m \max_{||x||=1} \sum_{k=1}^{m} x^T \Gamma_k x \tag{A.84}$$

$$\leq m \sum_{k=1}^{m} \max_{||x||=1} x^T \Gamma_k x = m \sum_{k=1}^{m} \bar{\sigma}(\Gamma_k) \tag{A.85}$$

$$= m \sum_{k=1}^{m} \max_{i} \bar{\sigma}^2(X_{ki}) \leq m^2 \max_{i,k} \bar{\sigma}^2(X_{ki}) \tag{A.86}$$

Here, (A.82) follows by the partitioning in (A.77); equation (A.83) follows from inequality (A.81); equation (A.84) follows by the definition of Γ_k in (A.79); equation (A.85) follows by maximizing separately over each term in the summation; and (A, 86) follows by the block diagonal structure of (A .79), Taking the square-root of (A.86) gives the desired relation (A.78).

■

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